ON THE MECHANICAL BEHAVIOR OF PRESTRESSED GLASS-REINFORCED PLASTICS

M. S. Chirko

UDC 539.319+373

A system of equations describing the behavior of prestressed glass-reinforced plastics is obtained on the basis of a model of multicomponent medium. A concrete problem, concerned with the concentration of residual stresses in a plate of such material, is considered.

To describe prestressed glass-reinforced plastics, we proceed from a model of multicomponent medium [1].

Let us have a certain volume V bounded by a surface S. We use σ_{ij}, u_i, e_{ij} to denote the stresses referred to a unit area, the displacements, and the strains in the filler, and use π_{ij}, v_i, ε_{ij} to denote the stresses referred to a unit area, the displacements, and the strains in the reinforcement. We assume that only the surface forces F_i act on the surface S, while mass forces and moments are absent from the volume V. We decompose F into two components

$$F_i = F_i^{(1)} + F_i^{(2)} \tag{1}$$

where $F_i^{(1)}$ is the surface force acting on the filler, while $F_i^{(2)}$ acts on the reinforcement. Then for the surface forces we obtain

$$F_i^{(1)} = \sigma_{ji}\nu_j, \qquad F_i^{(2)} = \pi_{ji}\nu_j \tag{2}$$

where ν_j is the unit vector of a normal to S. The conditions of equilibrium of the volume V have the form

$$\int\limits_S F_i ds = 0, \quad \int\limits_S \varepsilon_{ijk} x_j F_k ds = 0 \tag{3}$$

where ε_{ijk} is the axial tensor of Levi-Civita, and x_j is the j-th component of the Cartesian coordinate system. From the first equation of (3) and (2) we have

$$\sigma_{ij,j} + \pi_{ij,j} = 0 \tag{4}$$

If we introduce the quantity π_i

$$\pi_i = \sigma_{ij,j} \tag{5}$$

then from (4) we obtain

$$\sigma_{ij,j} - \pi_i = 0, \qquad \pi_{ij,j} + \pi_i = 0 \tag{6}$$

which constitutes separate equations of equilibrium for the filler and the reinforcement. From the form of these equations we can conclude that π_i is the vector of the force characterizing the interaction of the filler and the reinforcement, referred to a unit volume.

From the second equation (3) it follows that

$$\varepsilon_{ijk}(\sigma_{ji} + \pi_{ji}) = 0 \tag{7}$$

Voronezh. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 205-207, November-December, 1971. Original article submitted April 15, 1971.

From (7) we see that the stress tensors in the general case are not symmetric. Analogously to (5) we can introduce

$$\Sigma_k = \varepsilon_{ijk}\sigma_{ji} \tag{8}$$

where Σ_k is the moment interaction referred to a unit volume.

If by $\sigma_{(ij)}$, $\pi_{(ij)}$ we denote the symmetric parts of stress tensors, then for them the equations of equilibrium (6)-(8) assume the form [2]

$$\sigma_{(mn),m} + {}^1/_2\varepsilon_{imn}\Sigma_{i,m} - \pi_n = 0 \tag{9}$$

$$\pi_{(mn),m} - {}^1/_2\varepsilon_{imn}\Sigma_{i,m} + \pi_n = 0 \tag{10}$$

We now give a concrete form to the properties of the filler and the reinforcement. Above we used σ_{ij}, π_{ij}, the stress tensors referred to a unit area. It is obvious that true stresses acting within the media must enter the rheological equations. We denote them by σ_{ij}^* and π_{ij}^*.

Let α be the fraction of an area occupied by the filler. It is easy to see that α does not depend on the orientation of the area, but in the general case can be a function of the coordinates (for example, in the case of reinforcing the material along the radii of the polar coordinate system by filaments of constant thickness). We then have

$$\sigma_{ij} = \alpha\sigma_{ij}^*, \qquad \pi_{ij} = (1-\alpha)\pi_{ij}^* \tag{11}$$

The filler is considered to be a polymerizing medium subject to the equation [3]

$$\begin{aligned} \sigma_{(ij)}^* = \Big\{ & E_1(\eta)\, e_{kk} - \int_0^t e_{kk}\, \varphi_1(\eta, t-\tau)\, d\tau - \int_1^\eta e_{kk}(\eta)\, dE_1(\eta) + \\ & + \int_1^\eta e_{kk} d\Big[\int_0^t \varphi_1(\eta, t-\tau)\, d\tau\Big]\Big\}\delta_{ij} + E_2(\eta)\, e_{ij} - \int_0^t e_{ij}\, \varphi_2(\eta, t-\tau)\, d\tau - \\ & - \int_1^\eta e_{ij}(\eta)\, dE_2(\eta) + \int_0^\eta e_{ij} d\Big[\int_0^t \varphi_2(\eta, t-\tau)\, d\tau\Big] + C(\eta)\,\delta_{ij} \end{aligned} \tag{12}$$

where η is the degree of polymerization, taken to be known from the chemical kinetics by means of a function of time; $C(\eta)$ is a quantity characterizing the shrinkage of the material during the time of polymerization; E_1 and E_2 are the moduli of elasticity; φ_1 and φ_2 are the memory functions; δ_{ij} is Kronecker's symbol.

After end of polymerization $\eta = \eta_{max}$ Eq. (12) is transformed into the usual viscoelasticity equation, while certain constants of addition will characterize the residual stresses which arise in the polymerization process.

We assume that the reinforcement constitutes either absolutely flexible filaments which sustain only tensile stresses directed parallel to the x_1 axis, or glass fabric which is arranged in layers parallel to the x_1x_2 plane, and also sustains only tensile forces along the fibers of the fabric. For the first case we have

$$\pi_{11}^* = E\varepsilon_{11}, \quad \pi_{22}^* = \pi_{33}^* = 0, \quad \pi_{(ij)}^* = 0 \quad \text{for} \quad i \neq j \tag{13}$$

for the second case we have

$$\pi_{11}^* = E\varepsilon_{11}, \quad \pi_{22}^* = E_3\varepsilon_{22}, \quad \pi_{33}^* = 0, \quad \pi_{(ij)}^* = 0 \quad \text{for} \quad i \neq j \tag{14}$$

i.e., Poisson's ratio for fibers is zero.

We assume that when glass-reinforced plastics are manufactured, the fibers of reinforcement are first stretched and are then covered by a polymerizing mass. As the polymerization proceeds, the degree of gluing of the filler and the reinforcement increases, and when one of the media is being deformed, the interactions π_i and Σ_i arise. These interactions depend on the relative displacement of the media which take place after the gluing process, and also on the degree of polymerization, while for relaxing bonds it depends on the rates of displacements

$$\pi_i = K_{ij}\,[u_j - (v_j - v_j^{(0)}), \eta, t] \tag{15}$$

$$\Sigma_i = L_{ij}\left[\varepsilon_{jmn}\left(u_{m,n} - v_{m,n} + v_{m,n}^{(0)}\right), \eta, t\right] \tag{16}$$

where $v_j^{(0)}$ are the initial displacements of the reinforcement up to the removal of the preliminary tension; K_{ij}, L_{ij} are operators. For an elastic bond, i.e., one not explicitly depending on time, expanding (15) and (16) in Taylor's series with respect to displacements and confining ourselves to small quantities of the first order, we obtain

$$\pi_i = K_1(\eta)[u_i - (v_i - v_i^{(0)})] + K_2(\eta)[u_1 - (v_1 - v_1^{(0)})]\,\delta_{i1} \tag{17}$$

$$\Sigma_i = L_1(\eta)\,\varepsilon_{imn}\left(u_{m,n} - v_{m,n} + v_{m,n}^{(0)}\right) + L_2(\eta)\,\varepsilon_{1mn}\left(u_{m,n} - v_{m,n} + v_{m,n}^{(0)}\right)\delta_{i1} \tag{18}$$

for the case of reinforcement by fibers in the x_1 direction. In the second case (reinforcement by a fabric) we obtain

$$\pi_i = K_3(\eta)\left[u_i - \left(v_i - v_i^{(0)}\right)\right] + K_4(\eta)\,\delta_{i1}\left[u_1 - \left(v_1 - v_1^{(0)}\right)\right] + K_5(\eta)\,\delta_{i2}\left[u_2 - \left(v_2 - v_2^{(0)}\right)\right] \tag{19}$$

$$\Sigma_i = L_3(\eta)\,\varepsilon_{imn}\left(u_{m,n} - v_{m,n} + v_{m,n}^{(0)}\right) + L_4(\eta)\,\varepsilon_{1mn}\left(u_{m,n} - v_{m,n} + v_{m,n}^{(0)}\right)\delta_{i1} + L_5(\eta)\,\varepsilon_{2mn}\left(u_{m,n} - v_{m,n} + v_{m,n}^{(0)}\right)\delta_{i2} \tag{20}$$

where $K_i(\eta)$ and $L_i(\eta)$ are functions of the degree of polymerization.

Thus adding the Cauchy relationships to Eqs. (9)-(13), (17), (18) or (14), (19), (20), for the tensors e_{ij} and ε_{ij} we obtain a closed system of equations which describes the behavior of polymerizing glass-reinforced plastics.

We consider the problem of determining the residual stresses which arise in the glass-reinforced plastic material while it is being manufactured. Let a plate of glass-reinforced plastic material, infinite in the x_2 direction, have the width $2d(-d \le x_1 \le d)$ and its reinforcement be provided by fibers in the x_1 direction. The fibers are preliminarily stretched ($v_1^{(0)} = \varkappa x_1$).

The polymerization takes place within a closed volume, i.e., there are no displacements within the filler ($u_i = 0$). After hardening we remove constraints from the reinforcement and the filler, i.e., we obtain the free surfaces $\sigma_{11} = \sigma_{12} = \pi_{11} = 0$ for $x_1 = \pm d$.

As was already stated, after hardening the filler constitutes a viscoelastic medium which can be regarded as elastic by applying the Volterra principle to it, i.e.,

$$\sigma_{(ij)}^{*} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} + C(\eta_{\max})\,\delta_{ij} \tag{21}$$

where λ and μ are integral operators. The solution of the system (9), (10), (13), (17), (21), for the boundary conditions considered, has the form

$$\begin{gathered}
u_1 = A\,\mathrm{sh}\,\gamma x_1, \qquad u_2 = 0 \\
v_1 = -\frac{\lambda' + 2\mu'}{E'}A\,\mathrm{sh}\,\gamma x_1, \qquad \sigma_{12} = 0 \\
\sigma_{22} = \lambda'\left[A\gamma\,\mathrm{ch}\,\gamma x_1 - \frac{C - E'\varkappa}{E' + \lambda' + 2\mu'}\right] + C \\
\sigma_{11} = (\lambda' + 2\mu')A\gamma\,\mathrm{ch}\,\gamma x_1 + B, \qquad \pi_{11} = -(\lambda' + 2\mu')A\gamma\,\mathrm{ch}\,\gamma x_1 - B
\end{gathered}$$

where

$$\begin{gathered}
B = \frac{E'[(\lambda' + 2\mu')(\lambda' + 2\mu' - E')\varkappa + (\lambda' + 2\mu' + E')C]}{(\lambda' + 2\mu' + E')^2} \\
A = -\frac{B}{(\lambda' + 2\mu')\gamma\,\mathrm{ch}\,\gamma d}, \qquad \gamma = \sqrt{\frac{E' + \lambda' + 2\mu'}{E'(\lambda' + 2\mu')}(K_1 + K_2)} \\
E' = (1 - \alpha)E, \quad \lambda' = \alpha\lambda, \quad \mu' = \alpha\mu
\end{gathered} \tag{22}$$

From (22) we see that residual stresses will exist in the glass-reinforced plastic material. These stresses for an elastic glass-reinforced plastic material (λ and μ are constants and not operators) can be made equal to zero by choosing the preliminary tension in the form

$$\varkappa = -\frac{\lambda' + 2\mu' + E'}{(\lambda' + 2\mu')(\lambda' + 2\mu' - E')}C(\eta_{\max}) \tag{23}$$

since the filler volume decreases on hardening, i.e., $C(\eta)$ is negative.

LITERATURE CITED

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